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Application of variational principles to the axial extension of a circular cylindrical nonlinearly elastic membrane

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Abstract. In this paper stationary potential-energy and complementary-energy principles are formulated for boundary-value problems for compressible or incompressible nonlinearly elastic membranes, and full justification for adoption of the complementary principle is provided. The stationary principles are then extended to extremum principles, which provide upper and lower bounds on the energy functional associated with the solution of a given problem. The principles are then illustrated by their application to the nonlinear problem of the axially symmetric static deformation of an isotropic elastic membrane. In its undeformed natural configuration the membrane has the form of a circular cylindrical surface. The cylinder is subject to a prescribed (tensile) axial force with the ends of the cylinder constrained so that their radii remain constant. The alternative boundary condition in which the axial displacement of the ends is prescribed instead of the axial force is also considered.

The extremum principles are applied first without restriction on the form of strain-energy function in order to obtain primitive bounds on the energy of Voigt and Reuss type commonly used in composite-material mechanics. Then, for particular forms of strain-energy function, specific bounds are obtained by selecting suitable trial deformation and stress fields and the bounds are optimized using a numerical procedure (which is readily adapted for other forms of strain-energy function). It is found that these bounds are very close and hence give a good estimate of the actual energy. The associated deformed geometry of the membrane is described together with the resulting principal stresses.

Key words: nonlinear elasticity, elastic membranes, variational principles

1. Introduction

In this paper we consider a finite axially symmetric static deformation of an incompressible, isotropic elastic membrane whose undeformed natural reference configuration is a circular cylindrical surface. Specifically, we consider the deformation arising from *either* (a) a prescribed axial tensile force, or (b) prescribed axial displacement of the ends of the cylinder with, in each case, the radii of the cylinder ends held fixed. The inner and outer cylindrical surfaces are traction free. For problem (b) some numerical solutions were presented by Stoker [1] in respect of the Mooney–Rivlin form of strain-energy function, but there is, apparently, no other treatment of these problems available in the literature. However, we note that the problem of an axially extended membrane cylinder with internal pressure applied to ensure that the circular cylindrical shape is maintained has been discussed in detail by Haughton and

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Ogden [2] with particular reference to bifurcation from the circular cylindrical shape. We refer to this paper for citation of other relevant work.

For the problem at hand, as is commonly the case in nonlinear elasticity, the derivation of a closed-form solution would not seem to be possible, so that a numerical approach is required. Here, numerical solution is effected through the use of variational principles and associated extremum principles. Extensive discussion of variational principles for nonlinear elasticity is contained in Ogden [3, Chapter 5], for example, where problems associated with the construction of a true complementary energy principle are highlighted. Detailed references to the literature, which is concerned mostly with three-dimensional bodies, are given in [4]. The development and application of variational principles in nonlinear elasticity has also been considered by Lee and Shield [5, 6] and Haddow and Ogden [7]. Relatively little work has been set in the context of membrane theory, although application to a flat circular membrane under different boundary conditions has been discussed by Koiter [8], Lee and Shield [6] and Liu *et al.* [9] for specific forms of strain-energy function.

The problems mentioned in the first paragraph above provide motivation for a general nonlinearly elastic membrane formulation of complementary variational and extremum principles. Such a formulation is provided in this paper as the vehicle for obtaining approximate solutions for these and related membrane problems.

The basic equations are summarized in Section 2. Since the considered body is a membrane under tensile loading, a true complementary principle can be constructed unambiguously, as described in Section 3 for a general class of membrane problems. Conditions which guarantee the existence of a unique complementary energy function also ensure that the energy functional is minimized and the complementary energy functional is maximized for the (unique) actual solution of the problem. Details of this are given in respect of a general form of (incompressible, isotropic) strain-energy function. The stated conditions are exemplified in respect of both the neo-Hookean and Varga strain-energy functions, and in the latter case an explicit form of the complementary energy function is given. It is noted that the required conditions also hold for many commonly used forms of strain-energy function.

In Section 4 the specific problems for a cylindrical membrane are formulated and appropriate specializations of the energy and complementary energy are derived. Primitive bounds of Voigt and Reuss type (upper and lower bounds, respectively) on the energy functional associated with the actual solution are obtained by considering uniform admissible deformation and stress fields respectively without specialization of the strain-energy function.

The numerical approximation and optimization procedure is described in Section 5 for both the energy functional and the complementary energy functional. In the latter case we use a polynomial form of statically admissible stress field and the coefficients of the polynomial are chosen so as to maximize the complementary energy functional. This differs from the approach proposed by Lee and Shield [5], which was based on a statically admissible deformation gradient field.

In Section 6 numerical results are presented in respect of the neo-Hookean material and the closeness of the upper and lower bounds on the energy functional is demonstrated. The meridian curves obtained from the two principles are shown graphically and the variations of the principal stresses with the axial coordinate are also illustrated. The results confirm the positiveness of the principal stresses, which is fundamental to the validity of the analysis.

2. Basic equations

In formulating the stationary principles of energy and complementary energy and the associated extremum principles we follow the development described in Ogden [3, Chapter 5] and Haddow and Ogden [7], but with appropriate modification for the case of a membrane.

For a general treatment of the membrane equations and discussion of the membrane approximation appropriate to the present circumstances we refer to Haughton and Ogden [10], or, for specialization to the axially symmetric case, to Haughton and Ogden [2].

Let the membrane be identified with its middle surface $\mathcal{B}_0$ in its (undeformed, unstressed) natural configuration and let $\partial\mathcal{B}_0$ denote the bounding curve of $\mathcal{B}_0$. We take $\mathcal{B}_0$ as the reference configuration and let $\mathbf{X}$ denote the position vector of a point on $\mathcal{B}_0$ relative to some fixed origin. Let $\boldsymbol{\chi}$ denote the deformation and $\mathbf{x}$ the position vector of the material point $\mathbf{X}$ in the deformed configuration, so that

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) \quad \mathbf{X} \in \mathcal{B}_0, \tag{2.1}$$

and let $\mathcal{B}$ denote the image of $\mathcal{B}_0$ under the deformation.

The *surface deformation gradient* on $\mathcal{B}_0$, denoted $\mathbf{F}$, is defined as

$$\mathbf{F} = \text{Grad}\,\boldsymbol{\chi}, \tag{2.2}$$

where Grad is the surface gradient operator on $\mathcal{B}_0$. We may also expand $\mathbf{F}$ in the form

$$\mathbf{F} = \lambda_1 \mathbf{v}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda_2 \mathbf{v}^{(2)} \otimes \mathbf{u}^{(2)}, \tag{2.3}$$

where $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$ are the Lagrangian principal axes and $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ the Eulerian principal axes of the deformation of $\mathcal{B}_0$ (locally tangential to $\mathcal{B}_0$ and $\mathcal{B}$ respectively) and λ_1, λ_2 are the associated principal stretches.

We may extend (2.3) to the deformation gradient, denoted $\mathbf{A}$, of the bulk membrane material *evaluated on* $\mathcal{B}_0$ by writing

$$\mathbf{A} = \mathbf{F} + \lambda_3 \mathbf{v}^{(3)} \otimes \mathbf{u}^{(3)}, \tag{2.4}$$

where λ_3 is the stretch normal to the membrane, $\mathbf{u}^{(3)}$ is normal to the surface $\mathcal{B}_0$, taken in the sense that $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, $\mathbf{u}^{(3)}$ form a right-handed orthonormal triad, and $\mathbf{v}^{(3)}$ is the corresponding normal to $\mathcal{B}$, with $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\mathbf{v}^{(3)}$ also forming a right-handed orthonormal triad.

Analogously to (2.4) the nominal stress tensor $\mathbf{S}$ on $\mathcal{B}_0$ may be written

$$\mathbf{S} = \boldsymbol{\Sigma} + t_3 \mathbf{u}^{(3)} \otimes \mathbf{v}^{(3)}, \tag{2.5}$$

where $\boldsymbol{\Sigma}$ represents the stress in the surface $\mathcal{B}_0$ and t_3 the normal stress on $\mathcal{B}_0$.

The elastic properties of the membrane surface $\mathcal{B}_0$ may be characterized by use of the strain-energy function of the bulk material averaged through the thickness of the membrane in the reference configuration, as described by Haughton and Ogden [10]. The resulting energy function (per unit volume of the bulk material) may be regarded as a function of $\mathbf{A}$ and defined on $\mathcal{B}_0$, $W(\mathbf{A})$ say, with, for an incompressible material, the constraint

$$\det \mathbf{A} \equiv \lambda_1\lambda_2\lambda_3 = 1 \tag{2.6}$$

satisfied.

More specifically, for an isotropic material, to which attention is now confined, W depends on $\mathbf{A}$ only through the principal stretches (evaluated on $\mathcal{B}_0$). Thus, $W(\lambda_1, \lambda_2, \lambda_3)$, this dependence being indifferent to pairwise interchange of $\lambda_1, \lambda_2, \lambda_3$. The associated principal Biot stresses t_1, t_2, t_3 on $\mathcal{B}_0$ are given by

$$t_1 = \frac{\partial W}{\partial \lambda_1}, \qquad t_2 = \frac{\partial W}{\partial \lambda_2}, \qquad t_3 = \frac{\partial W}{\partial \lambda_3} \tag{2.7}$$

for an unconstrained material, and by

$$t_1 = \frac{\partial W}{\partial \lambda_1} - p\lambda_1^{-1}, \qquad t_2 = \frac{\partial W}{\partial \lambda_2} - p\lambda_2^{-1}, \qquad t_3 = \frac{\partial W}{\partial \lambda_3} - p\lambda_3^{-1} \tag{2.8}$$

for an incompressible material subject to (2.6).

For an isotropic material $\boldsymbol{\Sigma}$ may be written

$$\boldsymbol{\Sigma} = t_1\mathbf{u}^{(1)} \otimes \mathbf{v}^{(1)} + t_2\mathbf{u}^{(2)} \otimes \mathbf{v}^{(2)}. \tag{2.9}$$

We adopt the *membrane approximation*, which, in the present context, may be written simply as $t_3 = 0$. For an incompressible material we have $\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}$, while for an unconstrained material $t_3 = \partial W/\partial\lambda_3 = 0$ enables λ_3 to be expressed, at least implicitly, in terms of λ_1 and λ_2. In either case W may be treated as a function of λ_1 and λ_2 only and we write

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_3), \tag{2.10}$$

with λ_3 on the right-hand side replaced by the appropriate function of λ_1 and λ_2.

For either an incompressible or an unconstrained material it is then easy to see, using $t_3 = 0$, that

$$t_1 = \frac{\partial \hat{W}}{\partial \lambda_1}, \qquad t_2 = \frac{\partial \hat{W}}{\partial \lambda_2}. \tag{2.11}$$

Henceforth, we need not distinguish between unconstrained and incompressible materials and we characterize the material properties of the membrane through $\hat{W}(\lambda_1, \lambda_2) = \hat{W}(\lambda_2, \lambda_1)$. We may also regard $\hat{W}$ as a function of $\mathbf{F}$ and write $\hat{W}(\mathbf{F})$ to avoid duplication of notation. Then

$$\boldsymbol{\Sigma} = \frac{\partial \hat{W}}{\partial \mathbf{F}}. \tag{2.12}$$

When there are no tractions on the membrane surfaces the equations of equilibrium may be written in the form

$$\mathrm{Div}\,\boldsymbol{\Sigma} = \mathbf{0} \quad \text{on } \mathscr{B}_0, \tag{2.13}$$

where Div is the divergence operator on $\mathscr{B}_0$. Expressions for the components of the left-hand side of (2.13) may be found in, for example, Haughton and Ogden [10], but they are not needed here.

Equation (2.13) is coupled with edge conditions on $\partial\mathscr{B}_0$. Such conditions are exemplified by

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{X}) \quad \text{on } \partial\mathscr{B}_0^x, \tag{2.14}$$

$$\boldsymbol{\Sigma}^T\mathbf{N} = \boldsymbol{\sigma}(\mathbf{X}) \quad \text{on } \partial\mathscr{B}_0^\sigma, \tag{2.15}$$

where $\partial\mathscr{B}_0 = \partial\mathscr{B}_0^x \cup \partial\mathscr{B}_0^\sigma$, ^{T} denotes transpose, $\mathbf{N}$ is the unit outward normal to the edge $\partial\mathscr{B}_0$ lying (locally) in the tangent plane to $\mathscr{B}_0$ and $\boldsymbol{\xi}$ and $\boldsymbol{\sigma}$ are prescribed functions, the latter representing dead loading. Other types of edge loading may also be considered. In particular, we shall allow for the possibility that $\partial\mathscr{B}_0^x$ and $\partial\mathscr{B}_0^\sigma$ overlap in the sense that complementary components of $\boldsymbol{\xi}$ and $\boldsymbol{\sigma}$ are prescribed on $\partial\mathscr{B}_0^x \cap \partial\mathscr{B}_0^\sigma$.

We recall (Ogden [3, Chapter 6], [4]) that for an isotropic elastic material we have the polar decomposition $\mathbf{S} = \mathbf{T}\mathbf{R}^T$, where $\mathbf{T}$ is the Biot stress tensor and $\mathbf{R}$ is the rotation arising in the (unique) polar decomposition $\mathbf{A} = \mathbf{R}\mathbf{U}$ of the deformation gradient, where $\mathbf{U}$ is the (positive definite, symmetric) right stretch tensor. In the present context the appropriate specialization of this is

$$\boldsymbol{\Sigma} = \mathbf{T}\mathbf{R}^T, \tag{2.16}$$

and, for a given $\boldsymbol{\Sigma}$, the principal axes $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$ of $\mathbf{T}$ are determined from

$$\mathbf{T}^2 = \boldsymbol{\Sigma}\boldsymbol{\Sigma}^T, \tag{2.17}$$

where (with $t_3 = 0$) $\mathbf{T}$ has the spectral decomposition

$$\mathbf{T} = t_1\mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + t_2\mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} \tag{2.18}$$

and the rotation $\mathbf{R}$ may be expressed in the form

$$\mathbf{R} = \mathbf{v}^{(1)} \otimes \mathbf{u}^{(1)} + \mathbf{v}^{(2)} \otimes \mathbf{u}^{(2)}. \tag{2.19}$$

If attention is restricted to membrane stresses which are non-compressive, so that

$$t_1 \geqslant 0, \quad t_2 \geqslant 0, \tag{2.20}$$

then, for a given $\boldsymbol{\Sigma}$ such that $\det\boldsymbol{\Sigma} \equiv t_1t_2 \geqslant 0$ the polar decomposition (2.16) is unique. In fact, the strict versions of (2.20) are consequences of the strict local convexity of $\hat{W}(\mathbf{F})$ adopted in Section 3, which has important implications for the construction of a complementary energy function and for the validity of the principles of minimum energy and maximum complementary energy. The restrictions (2.20) ensure that wrinkling of the membrane is avoided (see, for example, Pipkin [11]).

3. Variational and extremum principles

3.1. Potential energy

For the considered membrane problem the *potential energy functional* may be written in the form

$$E\{\boldsymbol{\chi}\} = \int_{\mathcal{B}_0} \hat{W}(\lambda_1, \lambda_2)\mathrm{d}A - \int_{\partial\mathcal{B}_0^\sigma} \boldsymbol{\sigma} \cdot \boldsymbol{\chi}\,\mathrm{d}S, \tag{3.1}$$

where $\mathrm{d}A$ is the area element on $\mathcal{B}_0$ and $\mathrm{d}S$ is an element of arclength on $\partial\mathcal{B}_0$. In (3.1) we have taken the reference membrane thickness to be uniform and E to be defined *per unit membrane thickness*. In the integral over $\partial\mathcal{B}_0^{\boldsymbol{\sigma}}$ one, two or all three components of $\boldsymbol{\sigma}$ may be prescribed, but the complementary components of $\boldsymbol{\chi}$ are prescribed on $\partial\mathcal{B}_0^\sigma \cap \partial\mathcal{B}_0^x$ in the cases in which one or two components of $\boldsymbol{\sigma}$ are specified. If the traction is not prescribed on any part of $\partial\mathcal{B}_0$ then the integral over $\partial\mathcal{B}_0^\sigma$ is omitted from (3.1).

Let a superposed dot denote a variation in the quantity concerned. Then, considering variations in $\boldsymbol{\chi}$ subject to the kinematical boundary conditions, we have

$$\dot{\hat{W}} = t_1\dot{\lambda}_1 + t_2\dot{\lambda}_2 = \mathrm{tr}(\boldsymbol{\Sigma}\dot{\mathbf{F}}), \tag{3.2}$$

where use has been made of (2.3), (2.9) and (2.11).

On using (3.2) and standard manipulations we may write the first variation of E in the form

$$\dot{E} = -\int_{\mathcal{B}_0} (\mathrm{Div}\,\boldsymbol{\Sigma}) \cdot \dot{\boldsymbol{\chi}}\,\mathrm{d}A + \int_{\partial\mathcal{B}_0^\sigma} (\boldsymbol{\Sigma}^T\mathbf{N} - \boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\chi}}\,\mathrm{d}S. \tag{3.3}$$

Admissible variations $\dot{\boldsymbol{\chi}}$ are taken to be continuously differentiable in $\mathcal{B}_0$ and continuous on $\partial\mathcal{B}_0$. Since $\dot{\boldsymbol{\chi}}$ is an arbitrary admissible variation, it follows from the principle of virtual work that $\dot{E} = 0$ leads to Equations (2.13) and (2.15). Thus, we have the *stationary energy principle*, which states that, within the class of kinematically admissible variations, E is stationary if and only if $\boldsymbol{\chi}$ is a solution of the boundary-value problem defined by (2.12)–(2.15) with (2.2).

For an actual solution the second variation of E, denoted $\ddot{E}$ may be obtained, after some manipulation, in the form

$$\ddot{E} = \int_{\mathcal{B}_0} \mathrm{tr}\,(\dot{\boldsymbol{\Sigma}}\dot{\mathbf{F}})\mathrm{d}\mathbf{A}, \tag{3.4}$$

where $\dot{\boldsymbol{\Sigma}}$ is the variation in $\boldsymbol{\Sigma}$ induced by that in $\boldsymbol{\chi}$. An expression for $\mathrm{tr}\,(\dot{\boldsymbol{\Sigma}}\dot{\mathbf{F}})$ may be deduced from its counterpart for bulk solids given in Ogden [3, p. 449] for incompressible materials. Alternatively, if (2.3) and (2.9) are used it may be calculated directly in the form

$$\begin{aligned}\mathrm{tr}\,(\dot{\boldsymbol{\Sigma}}\dot{\mathbf{F}}) \;=\; & \lambda_1^2\hat{W}_{11}\eta_{11}^2 + 2\lambda_1\lambda_2\hat{W}_{12}\eta_{11}\eta_{22} + \lambda_2^2\hat{W}_{22}\eta_{22}^2 + \frac{\lambda_1\hat{W}_1 - \lambda_2\hat{W}_2}{\lambda_1^2 - \lambda_2^2}(\lambda_1^2\eta_{21}^2 + \lambda_2^2\eta_{12}^2)\\ & + 2\frac{\lambda_2\hat{W}_1 - \lambda_1\hat{W}_2}{\lambda_1^2 - \lambda_2^2}\lambda_1\lambda_2\eta_{12}\eta_{21} + \lambda_1\hat{W}_1\eta_{31}^2 + \lambda_2\hat{W}_2\eta_{32}^2,\end{aligned} \tag{3.5}$$

where $\hat{W}_1 = \partial\hat{W}/\partial\lambda_1$, $\hat{W}_{11} = \partial^2\hat{W}/\partial\lambda_1^2$, etc. and η_{ij} are the components of $\dot{\mathbf{F}}\mathbf{F}^{-1}$.

Necessary and sufficient conditions for the inequality tr $(\dot{\boldsymbol{\Sigma}}\dot{\mathbf{F}}) > \mathbf{0}$ to hold are

$$\begin{pmatrix} \hat{W}_{11} & \hat{W}_{12} \\ \hat{W}_{12} & \hat{W}_{22} \end{pmatrix} \quad \text{is positive definite}, \tag{3.6}$$

$$\frac{\hat{W}_1 - \hat{W}_2}{\lambda_1 - \lambda_2} > 0, \tag{3.7}$$

$$\hat{W}_1 > 0, \qquad \hat{W}_2 > 0, \tag{3.8}$$

jointly. Note that (3.7) and (3.8) together imply that

$$\frac{\lambda_1 \hat{W}_1 - \lambda_2 \hat{W}_2}{\lambda_1^2 - \lambda_2^2} > 0. \tag{3.9}$$

The condition (3.6) is a statement of strict local convexity of $\hat{W}(\lambda_1, \lambda_2)$. If (3.6) holds for all λ_1 and λ_2 then (3.7) follows. The additional inequalities (3.8) then ensure that $\hat{W}$ is (locally) strictly convex as a function of **F**. For convexity rather than strict convexity positive definiteness is replaced by positive semi-definiteness in (3.6) and $>$ by $\geqslant$ in (3.7) and (3.8), in which case the results were obtained by Pipkin [11]. We emphasize that these results are valid for either unconstrained or incompressible materials.

If (3.6)–(3.8) hold, then $\ddot{E} > 0$ for all admissible variations and the energy functional E associated with an actual solution (assuming such a solution exists) is a *local minimum*. Further, we emphasize that if the region of (λ_1, λ_2)-space for which (3.6)–(3.8) hold is convex then it follows that $\hat{W}$ is *globally* strictly convex as a function of **F**. Under these circumstances the actual solution $\boldsymbol{\chi}$ is unique and provides a *global minimum* of the energy. Moreover, the stress-deformation relation (2.12) is then uniquely invertible. Thus, we have the *extremum principle of minimum potential energy* in the form

$$E\{\boldsymbol{\chi}^*\} \geqslant E\{\boldsymbol{\chi}\} \tag{3.10}$$

for all admissible deformation fields $\boldsymbol{\chi}^*$, where $\boldsymbol{\chi}$ is the actual solution and equality holds if and only if $\boldsymbol{\chi}^* \equiv \boldsymbol{\chi}$. Admissible fields are taken to be twice-continuously differentiable deformations $\boldsymbol{\chi}^*$ which satisfy the prescribed conditions on $\partial\mathcal{B}_0^x$ and $\partial\mathcal{B}_0^\sigma \cap \partial\mathcal{B}_0^x$. Henceforth, we assume that the conditions for the validity of (3.10) are met, and in Section 3.3 we demonstrate that two common forms of strain-energy function do indeed satisfy the conditions for this to be the case.

The explicit form of $E\{\boldsymbol{\chi}^*\}$ in (3.10) is

$$E\{\boldsymbol{\chi}^*\} = \int_{\mathcal{B}_0} \hat{W}(\lambda_1^*, \lambda_2^*)\mathrm{d}A - \int_{\partial\mathcal{B}_0^\sigma} \boldsymbol{\sigma} \cdot \boldsymbol{\chi}^* \,\mathrm{d}S, \tag{3.11}$$

where λ_1^*, λ_2^* are the principal stretches calculated from $\boldsymbol{\chi}^*$.

3.2. Complementary energy

For an actual solution Equation (3.1) may be re-written in the form

$$E\{\boldsymbol{\chi}\} = \int_{\partial\mathcal{B}_0^x} (\boldsymbol{\Sigma}^T\mathbf{N}) \cdot \boldsymbol{\xi}\, \mathrm{d}S - \int_{\mathcal{B}_0} [\mathrm{tr}(\boldsymbol{\Sigma}\mathbf{F}) - \hat{W}(\mathbf{F})]\mathrm{d}A, \tag{3.12}$$

where $\boldsymbol{\Sigma}$ is the actual stress field associated with $\boldsymbol{\chi}$, with $\boldsymbol{\xi}$ (or some of its components) prescribed on $\partial\mathcal{B}_0^x$.

The integrand of the integral over $\mathcal{B}_0$ in (3.12) may be written

$$\mathrm{tr}(\boldsymbol{\Sigma}\mathbf{F}) - \hat{W}(\mathbf{F}) = \lambda_1 t_1 + \lambda_2 t_2 - \hat{W}(\lambda_1, \lambda_2). \tag{3.13}$$

Under the assumptions adopted in Section 3.1 strict convexity of $\hat{W}(\lambda_1, \lambda_2)$ is guaranteed and ensures the existence of a unique Legendre transform $\hat{W}_c(t_1, t_2)$ given by

$$\hat{W}_c(t_1, t_2) = \lambda_1 t_1 + \lambda_2 t_2 - \hat{W}(\lambda_1, \lambda_2), \tag{3.14}$$

interpreted as the *complementary energy per unit reference volume*. Moreover, the inverse of the stress-stretch relations (2.11) is unique and can be written

$$\lambda_1 = \frac{\partial \hat{W}_c}{\partial t_1}, \qquad \lambda_2 = \frac{\partial \hat{W}_c}{\partial t_2}. \tag{3.15}$$

As noted above, for a given nominal stress $\boldsymbol{\Sigma}$ a unique Biot stress $\mathbf{T}$ with $t_3 = 0$ and $t_1 \geqslant 0, t_2 \geqslant 0$ is defined and this is consistent with (3.8). Using (3.13) and (3.14) we may therefore regard the right-hand side of (3.12) as a functional of $\boldsymbol{\Sigma}$ and write

$$E_c\{\boldsymbol{\Sigma}\} = \int_{\partial\mathcal{B}_0^x} (\boldsymbol{\Sigma}^T\mathbf{N}) \cdot \boldsymbol{\xi}\mathrm{d}S - \int_{\mathcal{B}_0} \hat{W}_c(t_1, t_2)\mathrm{d}A, \tag{3.16}$$

which defines the complementary energy functional for the actual solution.

For admissible variations $\dot{\boldsymbol{\Sigma}}$, continuously differentiable and satisfying the equilibrium equation and stress boundary condition the first variation of $\hat{W}_c$ may be written

$$\dot{\hat{W}}_c = \lambda_1 \dot{t}_1 + \lambda_2 \dot{t}_2 = \mathrm{tr}(\mathbf{F}\dot{\boldsymbol{\Sigma}}), \tag{3.17}$$

analogously to (3.2).

Taking the first variation of (3.16) and using (3.17) followed by application of the divergence theorem we obtain

$$\dot{E}_c = \int_{\mathcal{B}_0} \mathrm{tr}[(\mathrm{Grad}\, \boldsymbol{\chi} - \mathbf{F})\dot{\boldsymbol{\Sigma}}]\mathrm{d}A, \tag{3.18}$$

where $\boldsymbol{\chi}$ is a deformation function satisfying the placement boundary condition. This provides the *complementary variational principle* for an isotropic elastic membrane. It states that, within the class of statically admissible variations, E_c is stationary if and only if $\boldsymbol{\Sigma}$ is a solution of the boundary-value problem, with deformation gradient $\mathbf{F}$, as constructed above, being the gradient of a deformation $\boldsymbol{\chi}$ satisfying the kinematical boundary conditions. Note, however, that the 'only if' part does not follow directly from (3.18) since the variations $\dot{\boldsymbol{\Sigma}}$, being divergence free, are not entirely arbitrary. The result is reached indirectly through the use of stress functions in a standard way.

Analogously to the procedure for calculating $\ddot{E}$, we calculate the second variation $\ddot{E}_c$ for an actual solution. This leads to

$$\ddot{E}_c = -\int_{\mathcal{B}_0} \operatorname{tr}(\dot{\boldsymbol{\Sigma}}\dot{\mathbf{F}})\mathrm{d}A, \tag{3.19}$$

where $\dot{\mathbf{F}}$ is the variation in $\mathbf{F}$ induced by that in $\boldsymbol{\Sigma}$.

Thus, under the inequalities (3.6)–(3.8) which ensure that $\ddot{E} > 0$ we have $\ddot{E}_c < 0$. Therefore, for an actual solution the complementary energy is a *local maximum*. Under the conditions which ensure that $E\{\boldsymbol{\chi}\}$ is a global minimum for the actual solution $\boldsymbol{\chi}$, it follows that $E_c\{\boldsymbol{\Sigma}\}$ is a *global maximum* of the complementary energy for the actual stress field $\boldsymbol{\Sigma}$.

For any statically admissible stress field $\boldsymbol{\Sigma}^*$, that is $\boldsymbol{\Sigma}^*$ satisfying the equilibrium Equation (2.13) and the boundary condition (2.15) with $\boldsymbol{\Sigma}^*$ continuously differentiable we define

$$E_c\{\boldsymbol{\Sigma}^*\} = \int_{\partial\mathcal{B}_0^x} (\boldsymbol{\Sigma}^{*T}\mathbf{N})\cdot\boldsymbol{\xi}\,\mathrm{d}S - \int_{\mathcal{B}_0} \hat{W}_c(t_1^*, t_2^*)\mathrm{d}A, \tag{3.20}$$

where t_1^*, t_2^* are the principal Biot stresses associated with $\boldsymbol{\Sigma}^*$ for which $t_1^* \geqslant 0, t_2^* \geqslant 0$ (with $t_3^* = 0$). Then, the *principle of maximum complementary energy* is stated in the form

$$E_c\{\boldsymbol{\Sigma}\} \geqslant E_c\{\boldsymbol{\Sigma}^*\}, \tag{3.21}$$

with equality holding if and only if $\boldsymbol{\Sigma}^* \equiv \boldsymbol{\Sigma}$, the actual stress field.

Combining (3.10) and (3.21) we then have

$$E\{\boldsymbol{\chi}^*\} \geqslant E\{\boldsymbol{\chi}\} = E_c\{\boldsymbol{\Sigma}\} \geqslant E_c\{\boldsymbol{\Sigma}^*\} \tag{3.22}$$

for all kinematically admissible $\boldsymbol{\chi}^*$ and statically admissible $\boldsymbol{\Sigma}^*$. This provides both upper and lower bounds on the energy functional associated with the actual solution.

3.3. Strain-energy functions

We now justify adoption of the inequalities (3.6)–(3.8) by illustrating that (3.6) and (3.7) hold for all λ_1 and λ_2 in respect of two commonly used forms of strain-energy function. We also note that in each case the inequalities $t_1 \geqslant 0, t_2 \geqslant 0$ define a convex region in (λ_1, λ_2)-space described by

$$\lambda_1^2\lambda_2 \geqslant 1, \quad \lambda_1\lambda_2^2 \geqslant 1. \tag{3.23}$$

3.3.1. *The neo-Hookean strain-energy function*

The (incompressible) neo-Hookean strain-energy function has the form

$$W = \tfrac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \tag{3.24}$$

where $\mu\ (> 0)$ is the shear modulus of the material in its stress-free natural configuration, and hence, on use of (2.6),

$$\hat{W}(\lambda_1, \lambda_2) = \tfrac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3). \tag{3.25}$$

It follows that

$$t_1 = \hat{W}_1 = \mu(\lambda_1 - \lambda_1^{-3}\lambda_2^{-2}), \qquad t_2 = \hat{W}_2 = \mu(\lambda_2 - \lambda_1^{-2}\lambda_2^{-3}) \tag{3.26}$$

and we deduce that $t_1 \geqslant 0$, $t_2 \geqslant 0$ if and only if (3.23) hold. We also have

$$\hat{W}_{11} = \mu(1 + 3\lambda_1^{-4}\lambda_2^{-2}),$$

$$\hat{W}_{11}\hat{W}_{22} - \hat{W}_{12}^2 = \mu^2(1 + 3\lambda_1^{-4}\lambda_2^{-2} + 3\lambda_1^{-2}\lambda_2^{-4} + 5\lambda_1^{-6}\lambda_2^{-6}),$$

which are both positive, and hence (3.6) holds. Next, we note that

$$\frac{\hat{W}_1 - \hat{W}_2}{\lambda_1 - \lambda_2} = \mu(1 + \lambda_1^{-3}\lambda_2^{-3}) > 0,$$

so that (3.7) follows. Under the conditions (3.23) the inequalities (3.8) hold except in the trivial case when there is no deformation. The region defined by (3.23) is depicted in Figure 1 as the unbounded (convex) region.

The stress-stretch relations (3.26) cannot be inverted explicitly to give λ_1 and λ_2 as functions of t_1 and t_2. Thus, the complementary energy (3.14) cannot be given explicitly in terms of t_1 and t_2. However, if we define

$$w_c(\lambda_1, \lambda_2) \equiv \hat{W}_c(t_1, t_2) = \lambda_1\hat{W}_1 + \lambda_2\hat{W}_2 - \hat{W} \tag{3.27}$$

then we obtain an explicit expression

$$w_c(\lambda_1, \lambda_2) = \tfrac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 - 5\lambda_1^{-2}\lambda_2^{-2} + 3) \tag{3.28}$$

for the complementary energy in terms of the principal stretches. For given values of t_1 and t_2 the corresponding values of λ_1 and λ_2 can then be obtained by numerical inversion of (3.26) in the implementation of the maximum complementary energy principle. This is done in the example considered in Section 6.

3.3.2. *The Varga strain-energy function*

The Varga form of (incompressible) strain-energy function is given by

$$W = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3), \tag{3.29}$$

and hence

$$\hat{W}(\lambda_1, \lambda_2) = 2\mu(\lambda_1 + \lambda_2 + \lambda_1^{-1}\lambda_2^{-1} - 3). \tag{3.30}$$

Thus,

$$t_1 = \hat{W}_1 = 2\mu(1 - \lambda_1^{-2}\lambda_2^{-1}), \qquad t_2 = \hat{W}_2 = 2\mu(1 - \lambda_1^{-1}\lambda_2^{-2}) \tag{3.31}$$

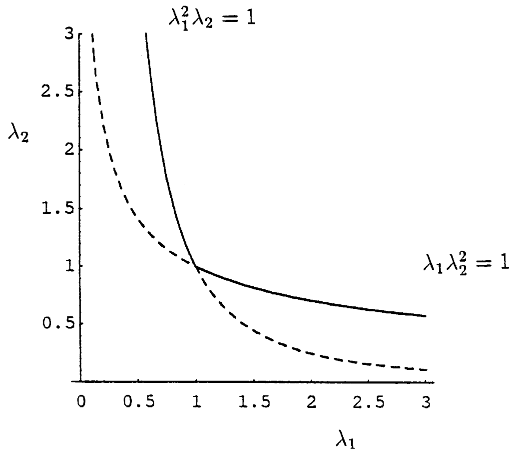

Figure 1. The region defined by (3.23) in (λ_1, λ_2)-space lies above and to the right of the continuous curves.

and it is easy to see that $t_1 \geqslant 0$, $t_2 \geqslant 0$ if and only if (3.23) hold. Also

$$\hat{W}_{11} = 4\mu\lambda_1^{-3}\lambda_2^{-1} > 0,$$

$$\hat{W}_{11}\hat{W}_{22} - \hat{W}_{12}^2 = 12\mu^2\lambda_1^{-4}\lambda_2^{-4} > 0,$$

$$\frac{\hat{W}_1 - \hat{W}_2}{\lambda_1 - \lambda_2} = 2\mu\lambda_1^{-2}\lambda_2^{-2} > 0,$$

so that (3.6) and (3.7) hold.

From the definition (3.14) and use of (3.31) the complementary energy function can be calculated explicitly as

$$\hat{W}_c(t_1, t_2) = 6\mu[1 - (1 - t_1/2\mu)^{1/3}(1 - t_2/2\mu)^{1/3}], \tag{3.32}$$

and, from (3.15), the inverse of (3.31) may then be given in the form

$$\lambda_1 = (1 - t_1/2\mu)^{-2/3}(1 - t_2/2\mu)^{1/3}, \qquad \lambda_2 = (1 - t_1/2\mu)^{1/3}(1 - t_2/2\mu)^{-2/3}. \tag{3.33}$$

Note that it follows from (3.33) that for the stretches to be positive and bounded each of t_1 and t_2 should be less than the value 2μ.

The neo-Hookean and Varga forms of strain-energy function are special cases of the function

$$\hat{W}(\lambda_1, \lambda_2) = \mu(\lambda_1^\alpha + \lambda_2^\alpha + \lambda_1^{-\alpha}\lambda_2^{-\alpha} - 3)/\alpha, \tag{3.34}$$

which is a one-term specialization of the class of functions introduced by Ogden [12], where $\mu\alpha > 0$. It is easy to show that this satisfies all the required conditions for validity of the theory provided $\alpha \geqslant 1$. With appropriate restrictions on the parameters the theory also applies to a wide range of other commonly used forms of strain-energy function.

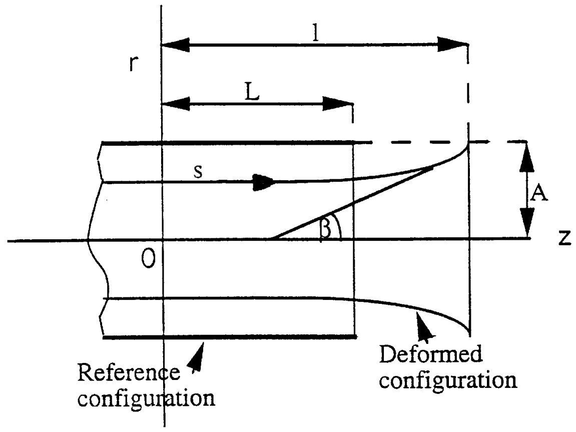

Figure 2. Sketch of the meridional section of the cylindrical membrane showing reference and deformed configurations.

4. Axial extension of a circular cylindrical membrane

Let the reference circular cylindrical surface $\mathcal{B}_0$ of the undeformed unstressed membrane be defined by

$$R = A, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad -L \leqslant Z \leqslant L, \tag{4.1}$$

where (R, Θ, Z) are cylindrical polar coordinates, A is the (constant) radius of the cylinder and $2L$ is its length.

We consider an axially symmetric deformation of $\mathcal{B}_0$ given by

$$r = r(Z), \quad \theta = \Theta, \quad z = z(Z), \tag{4.2}$$

where (r, θ, z) are cylindrical polar coordinates in the deformed configuration. The curve which is the intersection of the deformed membrane surface and a meridian plane is shown schematically in Figure 2. By symmetry it suffices to consider only that part for which $0 \leqslant Z \leqslant L$. Henceforth we refer to this curve as the meridian curve. In Figure 2, s denotes the distance along the meridian curve of a material point with axial coordinate Z and β is the angle between the tangent to the curve and the axial direction, as indicated.

The principal stretches in the membrane surface are given by

$$\lambda_1 = \frac{\mathrm{d}s}{\mathrm{d}Z}, \qquad \lambda_2 = \frac{r}{A}, \tag{4.3}$$

and it is easily shown that

$$r' = \lambda_1 \sin\beta, \qquad z' = \lambda_1 \cos\beta \tag{4.4}$$

and hence

$$\lambda_1 = (r'^2 + z'^2)^{1/2}, \tag{4.5}$$

where a prime denotes differentiation with respect to Z.

Since the faces of the membrane are taken to be free of traction the membrane approximation and the axial symmetry dictate that $\Sigma_{\Theta\theta}$, Σ_{Zr}, Σ_{Zz} are the only non-vanishing components of the nominal stress tensor $\mathbf{\Sigma}$, and they depend only on Z. It follows from (2.17) that

$$t_1 = \left(\Sigma_{Zr}^2 + \Sigma_{Zz}^2\right)^{1/2}, \quad t_2 = \Sigma_{\Theta\theta}, \tag{4.6}$$

and we also note the connections

$$\Sigma_{Zr} = t_1 \sin\beta, \qquad \Sigma_{Zz} = t_1 \cos\beta, \tag{4.7}$$

which are analogous to (4.4).

The nontrivial equilibrium equations obtained by specialization of (2.13) are

$$\frac{\mathrm{d}}{\mathrm{d}Z}\Sigma_{Zr} - \frac{\Sigma_{\Theta\theta}}{A} = 0, \tag{4.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}Z}\Sigma_{Zz} = 0. \tag{4.9}$$

It follows from (4.9) that $\mathbf{\Sigma}_{Zz}$ is constant.

The equilibrium Equations (4.8) and (4.9) can easily be shown to be equivalent to equations given by Green and Adkins [3, Section 4.11] for the axially symmetric deformation of a membrane, but for the application of the complementary energy principle, in particular, Equations (4.8) and (4.9) are more useful. See, also, [1]. We note in passing that from the above equations it can be shown that $\hat{W} - \lambda_1 t_1$ is constant (independent of Z). For a general axisymmetric membrane this result was established by Pipkin [14].

We consider two slightly different sets of boundary conditions.

Problem 1. Boundary conditions are prescribed in the form

$$r = A \quad \text{on } Z = \pm L, \tag{4.10}$$

$$\Sigma_{Zz} = F/2\pi AH \equiv t \quad \text{on } Z = \pm L, \tag{4.11}$$

where F is the prescribed axial load, H is the reference thickness of the membrane and the notation t is defined therein. For this problem it follows that $\Sigma_{Zz} = t$ for all Z. The deformed half-length $l = z(L)$ of the cylinder must be obtained as part of the solution.

For the application of the variational and extremum principles to this problem it is convenient to define the energy and complementary energy functionals per unit reference volume and we write

$$\bar{E} = E/4\pi AL, \qquad \bar{E}_c = E_c/4\pi AL, \tag{4.12}$$

recalling that E and E_c are defined per unit membrane thickness. Then, when applied to the considered boundary-value problem, (3.1) and (3.16) respectively become

$$\bar{E}\{\boldsymbol{\chi}\} = \frac{1}{2L}\int_{-L}^{L} \hat{W}(\lambda_1, \lambda_2)\mathrm{d}Z - \frac{tl}{L}, \tag{4.13}$$

$$\bar{E}_c\{\boldsymbol{\Sigma}\} = \Sigma_{Zr}(L)\frac{A}{L} - \frac{1}{2L}\int_{-L}^{L} \hat{W}_c(t_1, t_2)\mathrm{d}Z, \tag{4.14}$$

on adoption of (4.12), where $\Sigma_{Zr}(L)$ is the value of the component Σ_{Zr} of the admissible stress $\boldsymbol{\Sigma}$ on the cylinder ends.

Problem 2. For this problem the boundary condition (4.10) is retained, while (4.11) is replaced by specification of the axial displacement in the form

$$z = \pm l \quad \text{on} \quad Z = \pm L. \tag{4.15}$$

We set

$$l = \lambda L \tag{4.16}$$

with $\lambda > 1$, thus defining the *overall* axial stretch of the cylinder.

The functionals (4.13) and (4.14) are replaced by

$$\bar{E}\{\boldsymbol{\chi}\} = \frac{1}{2L}\int_{-L}^{L} \hat{W}(\lambda_1, \lambda_2)\mathrm{d}Z, \tag{4.17}$$

$$\bar{E}_c\{\boldsymbol{\Sigma}\} = \Sigma_{Zz}(L)\lambda + \Sigma_{Zr}(L)\frac{A}{L} - \frac{1}{2L}\int_{-L}^{L} \hat{W}_c(t_1, t_2)\mathrm{d}Z, \tag{4.18}$$

where $\Sigma_{Zz}(L)$ and $\Sigma_{Zr}(L)$ are the values of the statically admissible stresses Σ_{Zz} and Σ_{Zr} on the ends of the cylinder, Σ_{Zz} being constant.

4.1. Primitive bounds

4.1.1. *Problem* 1

With reference to (3.22), to obtain an elementary upper bound on the energy functional we choose a kinematically admissible deformation field with $\lambda_1^* = \lambda^*, \lambda_2^* = 1$, corresponding to a uniform axial extension. Then, on using the form (4.13) of the energy functional, we obtain

$$\bar{E}^* = \hat{W}(\lambda^*, 1) - t\lambda^*, \tag{4.19}$$

where $\bar{E}^*$ is the energy associated with the chosen field.

Similarly, a lower bound is obtained by choosing a statically admissible stress field with $t_1^* = t$ (to satisfy the axial boundary condition) and $t_2^* = 0$, corresponding to a uniform uniaxial tension. In this case, (4.14) gives

$$\bar{E}_c^* = -\hat{W}_c(t, 0). \tag{4.20}$$

By (3.22), the actual energy $\bar{E}$, scaled in accordance with (4.12), is then subject to the bounds

$$\hat{W}(\lambda^*, 1) - t\lambda^* \geqslant \bar{E} \geqslant -\hat{W}_c(t, 0). \tag{4.21}$$

The lower bound in (4.21) may also be written, on use of the Legendre transform (3.14) appropriately specialized, as

$$\hat{W}(\lambda_1, \lambda_2) - \lambda_1 t, \tag{4.22}$$

where λ_1 and λ_2 are given as functions of t through

$$\lambda_1 = \frac{\partial \hat{W}_c}{\partial t_1}(t, 0), \qquad \lambda_2 = \frac{\partial \hat{W}_c}{\partial t_2}(t, 0). \tag{4.23}$$

The upper bound in (4.21) is optimized when λ^* is such that

$$\frac{\partial \hat{W}}{\partial \lambda_1}(\lambda^*, 1) = t. \tag{4.24}$$

Thus, both bounds in (4.21) can be expressed (implicitly in general) as functions of t and hence, in principle, as functions of the resulting overall stretch, defined by $\lambda = l/L$.

For the neo-Hookean form of strain-energy function this procedure leads to the bounds

$$\tfrac{1}{2}\mu(3\lambda^{*-2} - \lambda^{*2} - 2) \geqslant \bar{E} \geqslant \tfrac{1}{2}\mu(4\lambda_1^{-1} - \lambda_1^2 - 3), \tag{4.25}$$

where

$$\mu(\lambda^* - \lambda^{*-3}) = t = \mu(\lambda_1 - \lambda_1^{-2}). \tag{4.26}$$

For the Varga strain-energy function we obtain the more explicit bounds

$$4\mu[(1 - t/2\mu)^{1/2} - 1] \geqslant \bar{E} \geqslant 6\mu[(1 - t/2\mu)^{1/3} - 1]. \tag{4.27}$$

We take these no further for Problem 1, but for Problem 2 we illustrate the results in more detail. Numerical results based on the bounds (4.25) are presented in Section 6.

4.1.2. *Problem* 2

For this problem the bounds (4.21) are replaced by

$$\hat{W}(\lambda, 1) \geqslant \bar{E} \geqslant t^*\lambda - \hat{W}_c(t^*, 0), \tag{4.28}$$

where λ is the prescribed overall axial stretch and t^* is the admissible axial stress component.

The lower bound in (4.28) is optimized by taking t^* such that

$$\frac{\partial \hat{W}_c}{\partial t_1}(t^*, 0) = \lambda, \tag{4.29}$$

which (implicitly) gives t^* as a function of λ. Thus, both bounds in (4.28) are functions of λ.

For the Varga material we obtain

$$2\mu(\lambda + \lambda^{-1} - 2) \geqslant \bar{E} \geqslant 2\mu(\lambda + 2\lambda^{-1/2} - 3). \tag{4.30}$$

In non-dimensional form with the terms in (4.30) scaled by 2μ the upper and lower bounds are plotted for $\lambda \geqslant 1$ in Figure 3(a) and the associated stresses $t/2\mu$ are plotted in Figure 3(b). The

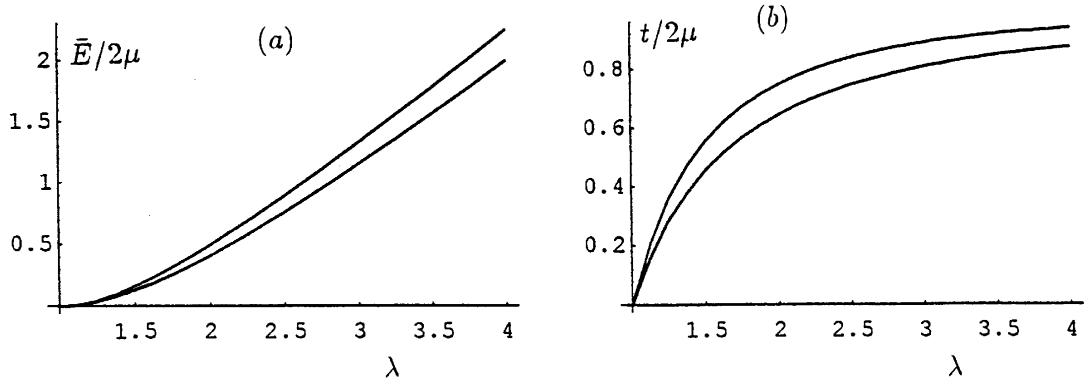

Figure 3. Plot of (a) $\bar{E}/2\mu$ and (b) $t/2\mu$ against λ in respect of the upper and lower bounds in (4.30).

upper and lower bounds are quite close together for this example considering the elementary nature of the chosen test fields.

5. Application of the extremum principles

5.1. Minimum energy principle

In order to implement the minimum energy principle we select an axially-symmetric deformation field, $\boldsymbol{\chi}^*$ say, with associated current cylindrical polar coordinates r^* and z^* expressed as polynomials in Z. For Problem 1, from the definition (4.13), we have

$$\bar{E}\{\boldsymbol{\chi}^*\} = \frac{1}{2L}\int_{-L}^{L} \hat{W}(\lambda_1^*, \lambda_2^*)\mathrm{d}Z - \frac{tl^*}{L}, \tag{5.1}$$

where $l^* = z^*(L)$, while for Problem 2 the latter term in (5.1) is omitted.

We adopt a non-dimensionalization in which $\hat{W}$, $\hat{W}_c$, $\bar{E}$, $\bar{E}_c$ and all stress components are scaled by division by the shear modulus μ and all lengths are scaled by division by A. The notation, however, is left unchanged by this scaling.

Thus, we require $r^*(\pm L) = 1$, $z^*(\pm L) = \pm l^*$. We choose r^* and z^* to have the forms

$$r^* = r_0^* + a_1 Z^2 + a_2 Z^4, \quad z^* = b_1 Z + b_2 Z^3, \tag{5.2}$$

where

$$a_2 = (1 - r_0^* - a_1 L^2)/L^4, \quad b_2 = (l^* - b_1 L)/L^3. \tag{5.3}$$

It follows that $\{l^*, r_0^*, a_1, b_1\}$ is a set of independent parameters and they are chosen, using a numerical procedure, so that the functional in (5.1), which is determined at discrete points, is minimized. The principal stretches λ_1^* and λ_2^* corresponding to $\boldsymbol{\chi}^*$ are obtained from the nondimensional forms of $(4.3)_2$ and (4.5), where the primes now signify differentiation with respect to nondimensional Z. The minimization yields an upper bound for the energy functional $\bar{E}$ and the resulting $\boldsymbol{\chi}^*$, corresponding to the parameters which minimize $\bar{E}\{\boldsymbol{\chi}^*\}$, is an approximate deformation field. An approximation $z^* = \zeta(r^*)$ to the equation of a meridian of the deformed membrane is then obtained from (5.2).

5.2. MAXIMUM COMPLEMENTARY ENERGY PRINCIPLE

To apply the complementary energy principle an axially-symmetric statically admissible stress field $\mathbf{\Sigma}^*$ is chosen with Σ^*_{Zr} and $\Sigma^*_{\Theta\theta}$ expressed as polynomials in Z and with Σ^*_{Zz} constant. Again, we use the nondimensionalization defined in Section 5.1 above. For Problem 2, from the definition (4.18), we have

$$\bar{E}_c\{\mathbf{\Sigma}^*\} = \Sigma^*_{Zz}\lambda + \Sigma^*_{Zr}(L)\frac{1}{L} - \frac{1}{2L}\int_{-L}^{L}\hat{W}_c(t_1^*, t_2^*)\mathrm{d}Z, \tag{5.4}$$

while for Problem 1 the term in λ is omitted.

A polynomial approximation which takes account of the symmetry of the problem and satisfies the equilibrium Equation (4.8) is given by

$$\Sigma^*_{Zr} = \sum_{i=0}^{n} c_{2i+1}Z^{2i+1}, \tag{5.5}$$

$$\Sigma^*_{\Theta\theta} = \sum_{i=1}^{n} (2i+1)c_{2i+1}Z^{2i}. \tag{5.6}$$

To evaluate the functional (5.4) for a given set of coefficients c_{2i+1}, $i \in \{0, 1, 2, \ldots, n\}$, t_1^* and t_2^* are found from (4.6), (5.5) and (5.6) and, for Problem 1, from the prescribed constant value $\Sigma_{Zz} = t$.

In general, difficulties arise in determining the form of the complementary energy function $\hat{W}_c(t_1^*, t_2^*)$. These are avoided, where necessary, by inverting the nondimensional form of (2.11) numerically in order to determine the stretches λ_1^*, λ_2^* corresponding to t_1^*, t_2^* at intervals of Z, and the values of $w_c(\lambda_1^*, \lambda_2^*)$, where w_c is defined by (3.27). The integral in (5.4) is then obtained by numerical integration. A numerical procedure is used to determine the set of coefficients c_{2i+1}, $i \in \{0, 1, 2, \ldots, n\}$, which maximizes (5.4), whose values are obtained at discrete points. The numerical results presented in Section 6 below are for $n = 3$ in (5.5) and (5.6).

The values λ_1^*, λ_2^* obtained from the stress field which maximizes (5.4) can be used to approximate the meridian curve in the deformed configuration. From the nondimensional forms of (4.3) and (4.5) we have

$$z^{*\prime} = (\lambda_1^{*2} - r^{*\prime 2})^{1/2}, \quad \lambda_2^* = r^*, \tag{5.7}$$

where r^* and z^* denote the values of r and z in the approximation. Since λ_2^* is known at increments of Z, then so is r^*, and $r^{*\prime}$ can then be found by numerical differentiation and substituted into the first equation in (5.7). Numerical integration of the values of $z^{*\prime}$ for discrete values of Z then enables r^* to be determined as a function of z^*.

6. Numerical results and discussion

Numerical results are presented for the neo-Hookean strain-energy function in the form (3.25) nondimensionalized by division by μ. The results are given for Problem 1 and are illustrated

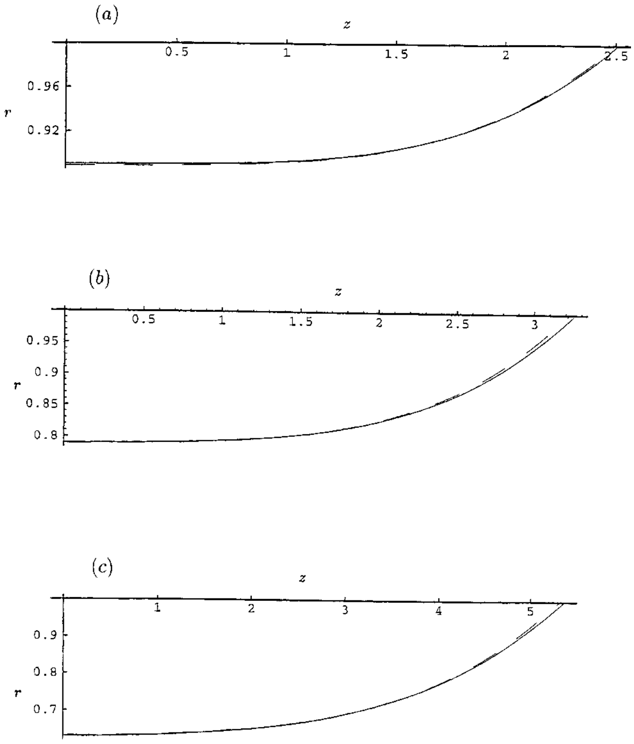

Figure 4. Plot of the meridian curve in the deformed configuration for $0 \leqslant Z \leqslant 2$. The dashed curves are obtained from the potential energy principle and the continuous curves from the complementary energy principle: (a) $F = 4$; (b) $F = 8$; (c) $F = 16$. The r scale is five times that for the z scale in (a) and four times in (b) and (c).

Table 1.

F	$\bar{E}^*$	$\bar{E}_c^*$	l^*	l_c^*	$\lambda_2^*(L)$
4	−0·71080	−0·71088	2·498	2·509	0·9998
8	−1·61452	−1·61456	3·223	3·250	0·9976
16	−4·29709	−4·29710	5·307	5·350	0·9981

for representative values $L = 2$ and $H = 0{\cdot}025$ of the dimensionless cylinder length and membrane thickness.

Upper and lower bounds $\bar{E}^*$ and $\bar{E}_c^*$ (scaled by division by μ) for the energy functional are given in Table 1 for three representative values of nondimensional F (defined as $F/\mu AH = 2\pi t/\mu$).

It is evident that the upper and lower bounds are very close. The deformed half-lengths l^* and l_c^* of the cylinder obtained from the energy and complementary energy principles, respectively, are also given in Table 1 along with the value of $\lambda_2^*(L)$ obtained from the complementary energy principle.

From the last column of Table 1 we see that the kinematical boundary condition (4.10), which is equivalent to $\lambda_2(\pm L) = 1$, is satisfied to a satisfactory level of approximation by the deformed configuration obtained from application of the complementary energy principle.

The meridian curves obtained from application of the two principles are shown graphically in Figure 4 for three values of F and it is evident that except near the ends the two principles

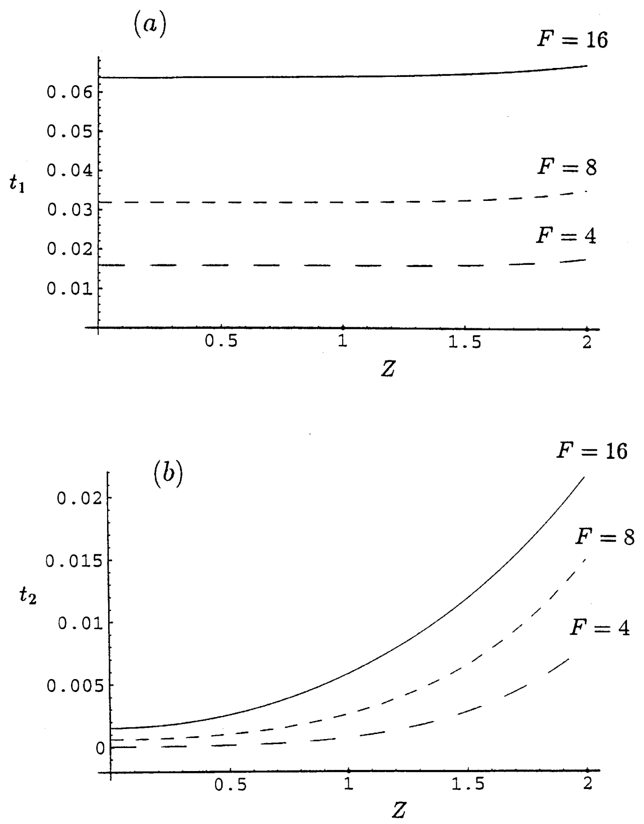

Figure 5. Plot of the principal Biot stresses for $0 \leqslant Z \leqslant 2$: (a) t_1; (b) t_2.

Table 2.

F	$\bar{E}^*$	$\bar{E}_c^*$	l^*	l_c^*
4	−0·6968	−0·7148	2·4126	2·5265
8	−1·4050	−1·6336	3·0892	3·2894
16	−4·1668	−4·3602	5·2063	5·3703

give almost identical results for the deformation, and even near the ends the difference is small. The difference has only been made apparent by an expansion of the scale used for r in the figures. The corresponding variations of the principal Biot stresses t_1, t_2 (nondimensionalized) with Z are plotted in Figure 5 and these plots confirm the positiveness of the principal stresses in accordance with the requirements (2.20).

Primitive bounds corresponding to uniform admissible fields have been discussed in Section 4.1. In respect of the neo-Hookean strain-energy function values of the upper and lower bounds on the energy and values of l^* and l_c^* for Problem 1 have been calculated on the basis of the discussion in Section 4.1.1. These are given in Table 2.

Clearly, as expected, the bounds shown in Table 2 are not as close as those given in Table 1, and they enclose those in Table 1.

The results discussed above are broadly similar to those obtained by Stoker [1], who used a finite-difference method to solve the governing equations for Problem 2. Direct numerical comparison is not possible, however, since Stoker's paper does not contain sufficient detail.

Acknowledgements

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References

1. J. J. Stoker, Elastic deformations of thin cylindrical sheets. *Progress in Applied Mechanics. The Prager Anniversary Volume*. New York: MacMillan Publishers (1963) pp.179–188.
2. D. M. Haughton and R. W. Ogden, Bifurcation of inflated circular cylinders of elastic material under axial loading I. Membrane theory for thin-walled tubes. *J. Mech. Phys. Solids* 27 (1979) 179–212.
3. R. W. Ogden, *Non-Linear Elastic Deformations*. Chichester: Ellis Horwood (1984) 532pp.
4. R. W. Ogden, Inequalities associated with the inversion of elastic stress-deformation relations and their implications. *Math. Proc. Cambridge Phil. Soc.* 81 (1977) 313–324.
5. S. J. Lee and R. T. Shield, Variational principles in finite elastostatics. *ZAMP* 31 (1980a) 437–453.
6. S. J. Lee and R. T. Shield, Application of variational principles in finite elasticity. *ZAMP* 31 (1980b) 454–472.
7. J. B. Haddow and R. W. Ogden, Compression of bonded elastic blocks. *J. Mech. Phys. Solids* 36 (1988) 551–579.
8. W. T. Koiter, On the complementary energy theorem in nonlinear elasticity theory. In: G. Fichera (ed.), *Trends in the Application of Pure Mathematics to Mechanics*. London: Pitman Publishers (1976) pp. 207–232.
9. S. Liu, J. B. Haddow and S. Dost, A variational approach to a circular hyperelastic membrane problem. *Acta Mech.* 99 (1993) 191–200.
10. D. M. Haughton and R. W. Ogden, On the incremental equations in nonlinear elasticity I. Membrane theory. *J. Mech. Phys. Solids* 26 (1978) 93–110.
11. A. C. Pipkin, The relaxed energy density for isotropic elastic membranes. *IMA J. Appl. Math.* 36 (1986) 85–99.
12. R. W. Ogden, Large deformation isotropic elasticity I: on the correlation of theory and experiment for incompressible rubberlike solids. *Proc. R. Soc. Lond. A* 326 (1972) 565–584.
13. A. E. Green and J. E. Adkins, *Large Elastic Deformations*. Oxford: Oxford University Press (1960) 348pp.
14. A. C. Pipkin, Integration of an equation in membrane theory. *ZAMP* 19 (1968) 818–819.